Risk Sensitive Shifted Rayleigh Filter for Underwater Bearings-Only Target Tracking Problems

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ABSTRACT

Tracking with bearings-only measurement is a challenging filtering problem since many years. It is observed that the angle measurements do not always adhere to the assumed statistics during the estimation interval. Sometimes, for a particular interval of time, an un-modeled measurement spike or exploding variance is observed, which corrupts the estimation accuracy severely. In this work, we develop a robust shifted Rayleigh filter (SRF) based on exponential quadratic cost criteria for a bearings-only tracking problem with such corrupted measurements. The performance of the proposed filter in terms of the root mean square error and the track-loss is compared with the SRF and the risk-sensitive unscented Kalman filter (RS-UKF). It has been observed that the proposed filter performs better than the RS-UKF and the SRF when measurement spike is present for a specific interval of time.

Keywords: Bearings-only tracking (BOT), shifted Rayleigh filter (SRF), risk-sensitive filtering, state estimation, measurement spike.

1. INTRODUCTION

Bearings-only tracking (BOT) is a non-linear filtering quest which has a wide range of applications, which includes aircraft surveillance, target tracking, navigation, to name a few.[1,2] BOT is highly significant in a war-like scenario where observer position is not to be revealed, thus asserting the use of only passive sensors. Therefore, the only information available to estimate kinematic states is noise corrupted bearings. The problem becomes challenging due to the unobservability of the system (without observer maneuver) and the non-linear measurement equation. The problem is often termed as target motion analysis since the primary objective is to estimate the position and velocity of the target.[1]

While working on real-life BOT scenarios, we found that the measurement does not strictly adhere to the assumed statistics. A 'sudden spike' or explosion of measurement variance is observed, which last for a certain number of iterations. One possible reasoning for such a reaction is that passive sensors mainly deal with acoustic positioning systems and due to the availability of multiple acoustic paths between source and observer, the measurements are affected and under such situations, traditional filters fail miserably. To mitigate this effect, we lean towards a class of robust filters known as the risk-sensitive filters. The risk-sensitive filters originated from the risk-sensitive control law optimize an exponential cost criterion to arrive at the estimator equations and is reported to be more robust in the presence of modeling error[3] and un-modeled bias in the system.[4] Since exponential quadratic cost criterion penalizes higher order moments, which were ignored in conventional quadratic cost, the risk-sensitive filters are more robust towards process and measurement uncertainties.[5] For estimation under such cost criteria, closed-form solution is available for linear Gaussian system, which is known as risk-sensitive Kalman filter.[3,6] For non-linear systems, no closed form solution exists and various forms of approximations based risk-sensitive filters have been developed for such systems. These approximate solutions include the risk-sensitive extended Kalman filter (RS-EKF),[7] the risk-sensitive unscented Kalman filter (RS-UKF),[8] the risk-sensitive particle filter,[9] adaptive grid risk-sensitive filter (AGRSF),[10] etc.

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It has been reported that the SRF is one of the best estimators when it comes to bearings-only tracking problems,[11].[12] It exploits the essential nature of non-linearity involved in BOT problems and calculates the exact posterior density using a Gaussian prior.[13] At the end of each iteration, this posterior is approximated as a Gaussian distribution to prepare for the next iteration, which accounts for the only approximation used in the SRF algorithm. We expect if a robust form of the SRF can be formulated, it will perform with more accuracy in presence of the process and measurement uncertainties. From this motivation, this work formulates a robust SRF, which uses the risk-sensitive cost criteria to deal with un-modeled measurement spike or the exploding variance as shown in Figure 1. The results are compared with the risk-sensitive unscented Kalman filter (RS-UKF) and the standard SRF. A huge track divergence is observed while using the risk-neutral filters under un-modeled measurement spike whereas a risk-sensitive filter has performed with a low failure rate. It has also been seen that the proposed risk-sensitive shifted Rayleigh filter (RS-SRF) is more accurate than the existing RS-UKF.

2. PROBLEM FORMULATION

The bearings-only tracking problem for a target moving with constant velocity is given by discrete time equation

$$\mathbf{x}_k = \mathbf{F}\mathbf{x}_{k-1} - \mho_{k-1,k} + \mathbf{w}_{k-1}, \tag{1}$$

where $\mathbf{x}_k$ is the relative state vector between the target and observer and is expressed as $[\mathrm{x_k}, \mathrm{y_k}, \mathrm{v_{x_k}}, \mathrm{v_{y_k}}]^T$. $\mathrm{x_k}$, $\mathrm{y_k}$ are the position components and $\mathrm{v_{y_k}}$, $\mathrm{v_{y_k}}$ are the velocity components of the relative state vector. $\mathbf{F}$ is the state transition matrix defined as

$$\mathbf{F} = \begin{bmatrix} I_{2\times2} & \Delta T I_{2\times2} \\ O_{2\times2} & I_{2\times2} \end{bmatrix},$$

$\mathbf{w}_{k-1}$ is a zero mean Gaussian distributed process noise with covariance matrix $\mathbf{Q}$. The covariance matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = \begin{bmatrix} \frac{\Delta T^3}{3} I_{2\times2} & \frac{\Delta T^2}{2} I_{2\times2} \\ \frac{\Delta T^2}{2} I_{2\times2} & \Delta T I_{2\times2} \end{bmatrix} \tilde{q},$$

where ΔT is the sampling time and $\tilde{q}$ denotes the process noise intensity.[14,15] $\mho_{k-1,k}$ defines a vector of deterministic inputs from observer with state vector $[\mathrm{x_k^o}, \mathrm{y_k^o}, \mathrm{v_{x_k}^o}, \mathrm{v_{y_k}^o}]^T$ and is given by[15]

$$\mho_{k-1,k} = \begin{bmatrix} \mathrm{x_k^o} - \mathrm{x_{k-1}^o} - \Delta T \mathrm{v_{x_{k-1}}^o} \\ \mathrm{y_k^o} - \mathrm{y_{k-1}^o} - \Delta T \mathrm{v_{y_{k-1}}^o} \\ \mathrm{v_{x_k}^o} - \mathrm{v_{x_{k-1}}^o} \\ \mathrm{v_{y_k}^o} - \mathrm{v_{y_{k-1}}^o} \end{bmatrix}.$$

The only measurement available in BOT scenario is the angle with respect to true north direction which is modeled by the equation

$$\theta_k = \tan^{-1}\left[\frac{\mathrm{x_k}}{\mathrm{y_k}}\right] + v_k, \tag{2}$$

where v_k denotes the measurement noise with probability density $v_k \sim \mathcal{N}(0, R)$.

2.1 Received Measurement

In this work, it is considered that the received measurements do not adhere with Eqn. (2) and a spike like measurement that lasts for a specific sampling time intervals is received as shown in Figure 1. As we do not use any real data, the following expressions are used to model the measurement spike.

$$\theta_i = \begin{cases} aA_b \frac{(i-t_s)}{\lfloor \frac{t_s+t_e}{2} \rfloor - t_s} + \theta_{t_s-1}; & \text{if } t_b < i \leq \lfloor \frac{t_s+t_e}{2} \rfloor, \\ -aA_b \frac{i-\lfloor \frac{t_s+t_e}{2} \rfloor}{t_e - \lfloor \frac{t_s+t_e}{2} \rfloor} + \theta_{\lfloor \frac{t_s+t_e}{2} \rfloor -1}; & \text{if } \lfloor \frac{t_s+t_e}{2} \rfloor < i \leq t_e, \\ \tan^{-1}\left[\frac{\mathrm{x_i}}{\mathrm{y_i}}\right] + v_k; & \text{else.} \end{cases} \tag{3}$$

Here, a can take values either of $+1$ and -1 with equal probabilities. Due to this we observe equal number of both rising and falling peaks for the spike. A_b is the bias amplitude, t_s and t_e denote the sampling time instants for spike start and end respectively. Note that the spike amplitude A_b and the time duration of the spike $t_e - t_s$ do not change and remain same for all Monte-Carlo runs.

3. RISK SENSITIVE SHIFTED RAYLEIGH FILTER

3.1 Augmented measurements

The shifted Rayleigh filter introduces an augmented measurement, $\boldsymbol{\alpha}_k$, which is the relative position vector between the observer and the target, and is given by[16]

$$\boldsymbol{\alpha}_k = \mathbf{H}\mathbf{x}_k + \boldsymbol{\gamma}_k. \tag{4}$$

In this equation, $\mathbf{H} = [\boldsymbol{I}_{r\times r}\ \boldsymbol{O}_{r\times(n-r)}]$, where r is the dimension of augmented measurement space, n is the order of the state vector ($n = 4$ in our case), and $\boldsymbol{I}_{r\times r}$ and $\boldsymbol{O}_{r\times r}$ are identity and zero matrix, respectively. We also define $\boldsymbol{\beta}_k = \Pi(\boldsymbol{\alpha}_k)$ as projection of $\boldsymbol{\alpha}_k$ onto a unit circle ($r = 2$) or a unit sphere ($r = 3$), as given in.[16] γ_k is the noise associated with augmented measurement model in (4) whose covariance $\mathbf{R_a}$ is expressed as follows[16]

$$\mathbf{R_a} = \sigma_\theta^2[\hat{\mathrm{x}}_{k|k-1}^2 + \hat{\mathrm{y}}_{k|k-1}^2 + var(\mathrm{x}_{k|k-1}) + var(\mathrm{y}_{k|k-1})]\boldsymbol{I}_{r\times r}, \tag{5}$$

where $var(.)$ denotes the covariance.

3.2 Solution approach

The cost function, which is optimized to formulate RS-SRF, is defined as[3]

$$\begin{aligned} J_k(\hat{\mathbf{x}}_k) &= \mathbb{E}\left[\exp\{\mu_1\sum_{i=0}^{k-1}(\mathbf{x}_i - \hat{\mathbf{x}}_i)^T(\mathbf{x}_i - \hat{\mathbf{x}}_i) + \mu_2(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T(\mathbf{x}_k - \hat{\mathbf{x}}_k)\}\right] \\ &= \int_{\mathbb{R}^n}\exp\left\{\mu_1\sum_{i=0}^{k-1}(\mathbf{x}_i - \hat{\mathbf{x}}_i)^T(\mathbf{x}_i - \hat{\mathbf{x}}_i) + \mu_2(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T(\mathbf{x}_k - \hat{\mathbf{x}}_k)\right\} p(\mathbf{x}_k|\boldsymbol{\alpha}_k)d\mathbf{x}_k. \end{aligned} \tag{6}$$

3.3 Risk-sensitive filtering with augmented measurements

We define an *information state*, σ_k, in order to work out the estimate recursively that minimizes the cost function in Eqn. (6). Information state is a function of available information set that completely summarizes the past of the system in a probabilistic sense.[2] One such candidate for the information state is the un-normalized conditional density function in risk sensitive sense, which is given by[3]

$$\sigma_k = \exp\left\{\mu_1\sum_{i=0}^{k-1}(\mathbf{x}_i - \hat{\mathbf{x}}_i)^T(\mathbf{x}_i - \hat{\mathbf{x}}_i)\right\} p(\mathbf{x}_k|\boldsymbol{\alpha}_k). \tag{7}$$

Substituting in (6), the cost function can now be compactly re-written as

$$J_k(\hat{\mathbf{x}}_k) = \int_{\mathbb{R}^n}\exp\left(\mu_2(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T(\mathbf{x}_k - \hat{\mathbf{x}}_k)\right)\sigma_k d\mathbf{x}_k, \tag{8}$$

and the minimum cost estimate is expressed as

$$\hat{\mathbf{x}}_k = \arg\min_{r\epsilon\mathbb{R}^n}\int\exp(\mu_2(\mathbf{x}_k - r)^T(\mathbf{x}_k - r))\sigma_k d\mathbf{x}_k. \tag{9}$$

We will exploit the linear property of the process equation and the augmented measurement in Eqn. (4) to derive the risk sensitive estimate. Since we have used the conditional pdf as the information state, it follows the recursion

$$\sigma_{k+1} = \int p(\boldsymbol{\alpha}_{k+1}|\boldsymbol{\alpha}_k, \mathbf{x}_{k+1})p(\mathbf{x}_{k+1}|\mathbf{x}_k)\sigma_k d\mathbf{x}_k. \tag{10}$$

Here, $p(\boldsymbol{\alpha}_{k+1}|\boldsymbol{\alpha}_k, \mathbf{x}_{k+1})$ is the distribution of predicted measurement at time $k+1$ given measurements upto time k. $p(\mathbf{x}_{k+1}|\mathbf{x}_k)$ denotes the prior state distribution, both of these densities can be found out if density of process noise $\mathbf{w}_k$ and measurement noise $\boldsymbol{\gamma}_k$ respectively are known.

The final risk sensitive equations for system defined by Eqn. (1) and Eqn. (4) can be obtained by assuming a Gaussian distributed conditional density σ_k, following the recursion given by Eqn. (10) and utilizing Eqn. (9) to find the minimum cost estimate.[3] The final risk sensitive equations are

$$\begin{aligned}
\hat{\mathbf{x}}_{k+1|k} &= \mathbf{F}\hat{\mathbf{x}}_{k|k} \\
\boldsymbol{\Sigma}_{k+1|k} &= \mathbf{F}[(\boldsymbol{\Sigma}_{k|k}^{-1} - 2\mu_1 I)^{-1}]\mathbf{F}^T + \mathbf{Q} \\
\hat{\mathbf{x}}_{k+1|k+1} &= \hat{\mathbf{x}}_{k+1|k} + \boldsymbol{\Sigma}_{k+1|k}\mathbf{H}^T(\mathbf{H}\Sigma_{k+1|k}\mathbf{H}^T + R)^{-1}[\boldsymbol{\alpha}_{k+1} - \mathbf{H}\hat{\mathbf{x}}_{k+1|k}] \\
\boldsymbol{\Sigma}_{k+1|k+1} &= (I - \boldsymbol{\Sigma}_{k+1|k}\mathbf{H}^T(\mathbf{H}\Sigma_{k+1|k}\mathbf{H}^T + \mathbf{R_a})^{-1}\mathbf{H})\boldsymbol{\Sigma}_{k+1|k},
\end{aligned} \tag{11}$$

where the term $\boldsymbol{\Sigma}_{k+1|k}\mathbf{H}^T(\mathbf{H}\boldsymbol{\Sigma}_{k+1|k}\mathbf{H}^T + \mathbf{R_a})^{-1}$ is referred to as the Kalman gain, $\mathbf{K}_{k+1}$.

3.4 Risk-sensitive shifted Rayleigh filter (RS-SRF)

The derivation for the risk-sensitive shifted Rayleigh filter will closely follow the approach given in[16, 17] as the expressions with augmented measurement are available in linear form. In the derivation of SRF equations, the assumption of direct accessibility of the augmented measurements is maintained. However, since the range is not available with us, filtering equations are altered by evaluating moments of $\mathbf{x}_k$ conditioned on $\boldsymbol{\beta}_k$. Writing $\boldsymbol{\alpha}_k = r_k\boldsymbol{\beta}_k$, where $r_k = ||\boldsymbol{\alpha}_k||$ and $\boldsymbol{\beta}_k$ is defined in section 3.1. If $\hat{\mathbf{x}}_{k|k}$ is approximated as a Gaussian distributed random variable, it is legit to write $\mathbf{x}_k$ as

$$\mathbf{x}_{k+1} = (I - \mathbf{K}_{k+1}\mathbf{H})\hat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}r_{k+1}\boldsymbol{\beta}_{k+1} + \boldsymbol{\xi}_{k+1}, \tag{12}$$

where $\boldsymbol{\xi}_{k+1}$ follows the probability density $\mathcal{N}(0, (I - \mathbf{K}_{k+1}\mathbf{H})\boldsymbol{\Sigma}_{k+1|k})$. Taking the conditional expectation of $\mathbf{x}_{k+1}$ given $\boldsymbol{\beta}_{k+1}$, we have

$$\begin{aligned}
\hat{\mathbf{x}}_{k+1|k+1} &= E(\mathbf{x}_{k+1}|\boldsymbol{\beta}_{k+1}) \\
&= (I - \mathbf{K}_{k+1}\mathbf{H})\hat{\mathbf{x}}_{k+1|k} + E[r_{k+1}|\boldsymbol{\beta}_{k+1}]\mathbf{K}_{k+1}\boldsymbol{\beta}_{k+1}.
\end{aligned} \tag{13}$$

Similarly, the posterior covariance is calculated as[13]

$$\begin{aligned}
cov(\mathbf{x}_{k+1}|\boldsymbol{\beta}_{k+1}) &= E[(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1})^T(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1})|\boldsymbol{\beta}_{k+1}] \\
&= (I - \mathbf{K}_{k+1}\mathbf{H})\boldsymbol{\Sigma}_{k+1|k} + cov[r_{k+1}|\boldsymbol{\beta}_{k+1}]\mathbf{K}_{k+1}\boldsymbol{\beta}_{k+1}\boldsymbol{\beta}_{k+1}^T\mathbf{K}_{k+1}^T.
\end{aligned} \tag{14}$$

Eqn. (13) and Eqn. (14) can be expanded to get the final equations for risk sensitive SRF[16] as follows:

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}\hat{\mathbf{x}}_{k-1|k-1} - \boldsymbol{\mho}_{k-1,k}, \tag{15}$$

$$\boldsymbol{\Sigma}_{k|k-1} = \mathbf{F}[(\boldsymbol{\Sigma}_{k-1|k-1}^{-1} - 2\mu_1 I)^{-1}]\mathbf{F}^T + \mathbf{R_a}, \tag{16}$$

$$\hat{\mathbf{x}}_{k|k} = (I - \mathbf{K}_k\mathbf{H})\hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k E[r_k|\boldsymbol{\beta}_k]\mathbf{V}_k^{1/2}\boldsymbol{\beta}_k, \tag{17}$$

$$\boldsymbol{\Sigma}_{k|k} = (I - \boldsymbol{\Sigma}_{k|k-1}\mathbf{H}^T(\mathbf{H}\boldsymbol{\Sigma}_{k|k-1}\mathbf{H}^T + \mathbf{R_a})^{-1}\mathbf{H})\boldsymbol{\Sigma}_{k|k-1} + cov(r_k|\boldsymbol{\beta}_k)\mathbf{K}_k\mathbf{V}_k^{-1/2}\boldsymbol{\beta}_k{\boldsymbol{\beta}_k}^T\mathbf{V}_k^{1/2}\mathbf{K}_k^T, \tag{18}$$

where $E[r_k|\boldsymbol{\beta}_k]$ and $cov[r_k|\boldsymbol{\beta}_k]$ are given in.[16]

4. SIMULATION RESULTS

A typical measurement spike is presented in Figure 1. The spike is included by modifying the mathematical model of measurement equation within a certain range of time steps as expressed by Eqn. (3). The process noise intensity $\tilde{q}$ is taken as[18] 2.142×10^{-6} km^2/min^3. The sampling time taken between successive measurements is 1 second. The risk-sensitive parameter μ_1 is considered to be -5 for this simulation work. Measurement spike amplitude is 15° and the observation period lasts for 30 minutes.

The values of all parameters used and the initialization of the different filters considered are according to the method given in.[18] Further, two metrics, the root mean square error (RMSE) and the track-loss, are defined, as illustrated in,[18] to compare the performance of the different filters considered in this simulation. The track-loss is considered to occur when $RMSE^{pos}_{k_m} > T_l$ km, where

$$RMSE^{pos}_{k_m} = \sqrt{(\mathrm{x}^t_{k_m} - \hat{\mathrm{x}}^t_{k_m})^2 + (\mathrm{y}^t_{k_m} - \hat{\mathrm{y}}^t_{k_m})^2},$$

k_m is the final time step and T_l is the threshold value, which is taken as 1 km for this work. Note that diverged tracks and corresponding Monte-Carlo runs are excluded while calculating the position RMSE.

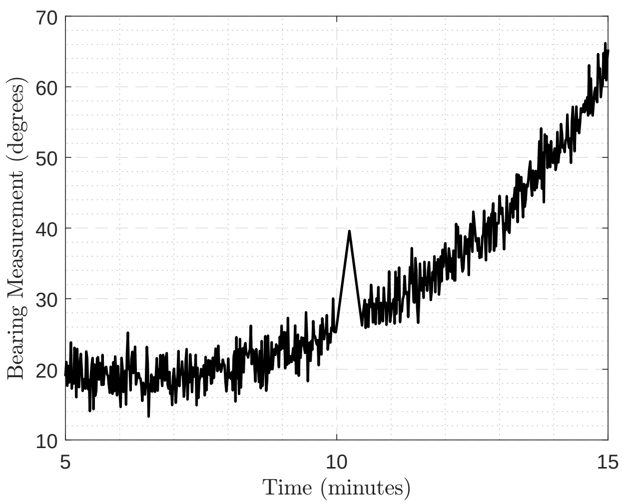

Figure 1: Measurement Spike

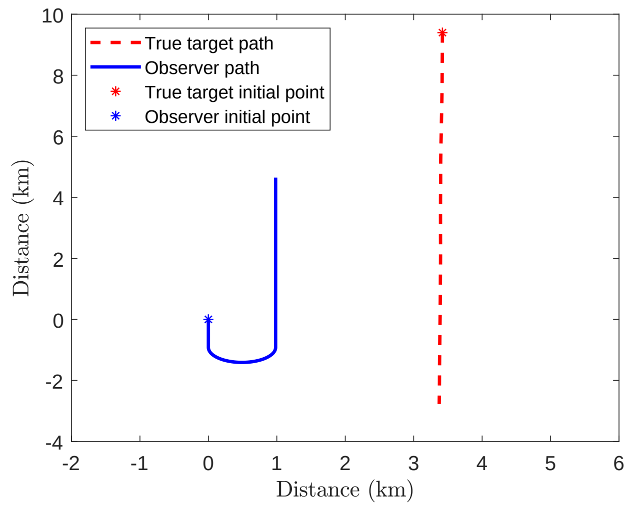

Figure 2: Observer and target dynamics

4.1 Scenario

We have chosen the RS-UKF[8] and the standard SRF for comparison. The BOT scenario considers a linear process model (constant velocity target) and is presented in Figure 2. The observer traverses with an initial course of 180^o with respect to true north at a velocity of 8 knots. From 1^{st} to 6^{th} minute the observer maneuvers from 180° to 0° at constant rate. The target moves with a constant velocity of 15 knots at a course of 180° from true north.

The measurement spike is shown in Figure 1 and is included between $t_s = 600$ and $t_e = 660$ sampling time instants for a total of 60 seconds. In this work, we have assumed that there is no ambiguity in its time of occurrence and maximum amplitude, thus for all Monte-Carlo runs, the glitch occurs at the same time. Maximum amplitude is taken as 15^o. Since the measurements were highly corrupted during measurement spike phase, we tried using only process update in Kalman filter recursions and using measurement update only when measurements are convincing. This also lead to considerable divergence while tracking.

On comparison with the RS-UKF, we found that it takes much more time to settle down than the RS-SRF even with low initialization errors. The final RMSE in position was better for the RS-SRF as shown in Figure 3. The results with track-loss and terminal RMSE in position and velocity are mentioned in Table 1. An appreciable difference in track-loss is evident from the table.

Result: It was observed that the risk neutral filters ultimately failed leading to 100% track-loss. With the robust approach a 10000 Monte Carlo run simulation cycle have shown 670 cases of track-loss in the RS-UKF whereas for the specified value of risk sensitive parameter, the same is 0 in case of RS-SRF. Table 1 shows the comparison between the RS-SRF, RS-UKF and SRF in terms of the track-loss, RMSE position and RMSE velocity. The RMSE position is plotted in Figure 3. Since the SRF and UKF lead to remarkably high track-loss, they are not presented in the figures. Superior performance of the RS-SRF in presence of measurement spike is evident over other mentioned filters.

Table 1: Terminal RMSE in position and velocity, and % track-loss for different filters

	$\mathbf{RMSE^{pos}_{k_m}}$(m)	$\mathbf{RMSE^{vel}_{k_m}}$(m/s)	Track-loss
RS-SRF	292	1.28	0.0%
RS-UKF	308	1.31	6.7%
SRF	–	–	100%

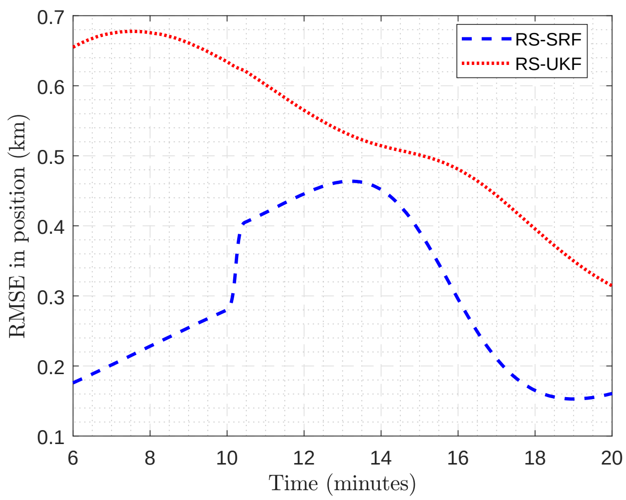

Figure 3: RMSE position

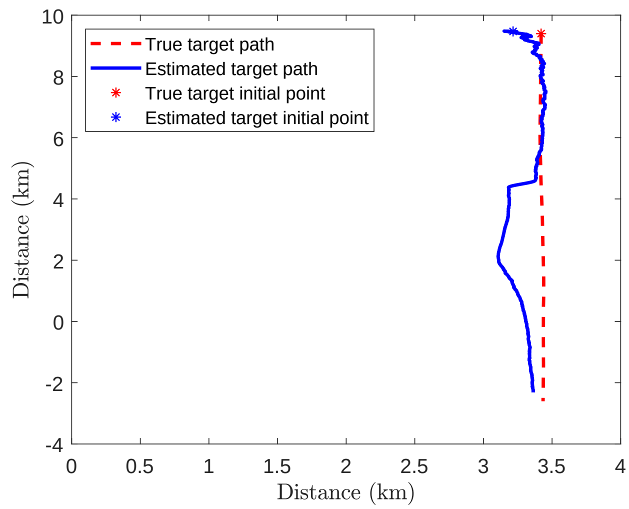

Figure 4: Truth and estimated target

5. CONCLUSIONS

Conventional moment matching filters have shown high track divergence while dealing with measurement spike and thus lose their applicability in underwater passive BOT systems. In this technical work, the shifted Rayleigh filter has been modified by switching from the existing quadratic error criterion to exponential quadratic error criterion. To validate the performance of proposed filter, an underwater real life BOT problem including measurement spike is solved using risk neutral SRF, RS-UKF and the proposed RS-SRF. The simulation results suggest the use of RS-SRF if measurement spike is likely to occur.

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